A. N. Parshin
I. R. Shafarevich
(Eds.)

Algebraic Geometry III

Complex Algebraic Varieties
Algebraic Curves and Their Jacobians

Springer

List of Editors, Authors and Translators

Editor-in-Chief

R.V. Gamkrelidze, Russian Academy of Sciences, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow; Institute for Scientific Information (VINITI), ul. Usievicha 20a, 125219 Moscow, Russia; e-mail: gam@ipsun.ras.ru

Consulting Editors

A. N. Parshin, I. R. Shafarevich, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow, Russia

Authors

Viktor S. Kulikov, Chair of Applied Mathematics II, Moscow State University of Transport Communications (MIIT), ul. Obraztcova 15, 101475 Moscow, Russia; e-mail: kulikov@alg.mian.su and victor@olya.ips.ras.ru

P.F. Kurchanov, Chair of Applied Mathematics II, Moscow State University of Transport Communications (MIIT), ul. Obraztcova 15, 101475 Moscow, Russia

V.V. Shokurov, Department of Mathematics, The Johns Hopkins University, Baltimore, MA 21218–2689, USA; e-mail: shokurov@chow.mat.jhu.edu

Translator

I. Rivin, Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom, e-mail: igor@maths.warwick.ac.uk; Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA, e-mail: rivin@caltech.edu

Contents

I. Complex Algebraic Varieties: Periods of Integrals and Hodge Structures

Vik. S. Kulikov, P. F. Kurchanov

Translated from the Russian
by Igor Rivin

Contents

Introduction

Starting with the end of the seventeenth century, one of the most interesting directions in mathematics (attracting the attention as J. Bernoulli, Euler, Jacobi, Legendre, Abel, among others) has been the study of integrals of the form

$$A_w(\tau) = \int_{\tau_0}^{\tau} \frac{dz}{w},$$

where w is an algebraic function of z. Such integrals are now called *abelian*.

Let us examine the simplest instance of an abelian integral, one where w is defined by the polynomial equation

$$w^2 = z^3 + pz + q, \tag{1}$$

where the polynomial on the right hand side has no multiple roots. In this case the function A_w is called an *elliptic integral.* The value of A_w is determined up to $m\nu_1 + n\nu_2$, where ν_1 and ν_2 are complex numbers, and m and n are integers. The set of linear combinations $m\nu_1 + n\nu_2$ forms a lattice $H \subset \mathbb{C}$, and so to each elliptic integral A_w we can associate the torus $\mathbb{C}/H$.

On the other hand, equation (1) defines a curve in the affine plane $\mathbb{C}^2 = \{(z,w)\}$. Let us complete $\mathbb{C}^2$ to the projective plane $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ by the addition of the "line at infinity", and let us also complete the curve defined by equation (1). The result will be a nonsingular closed curve $E \subset \mathbb{P}^2$ (which can also be viewed as a Riemann surface). Such a curve is called an *elliptic curve.*

It is a remarkable fact that the curve E and the torus $\mathbb{C}/H$ are isomorphic Riemann surfaces. The isomorphism can be given explicitly as follows.

Let $\wp(z)$ be the Weierstrass function associated to the lattice $H \subset \mathbb{C}$.

$$\wp = \frac{1}{z^2} + \sum_{h \in H, h \neq 0} \left[\frac{1}{(z-2h)^2} - \frac{1}{(2h)^2} \right].$$

It is known that $\wp(z)$ is a doubly periodic meromorphic function with the period lattice H. Further, the function $\wp(z)$ and its derivative $\wp'(z)$ are related as follows:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{2}$$

for certain constants g_2 and g_3 which depend on the lattice H. Therefore, the mapping $z \to (\wp(z), \wp'(z))$ is a meromorphic function of $\mathbb{C}/H$ onto the compactification $E' \subset \mathbb{P}^2$ of the curve defined by equation (2) in the affine plane. It turns out that this mapping is an isomorphism, and furthermore, the projective curves E and E' are isomorphic!

Let us explain this phenomenon in a more invariant fashion. The projection $(z,w) \to z$ of the affine curve defined by the equation (1) gives a double covering $\pi : E \to \mathbb{P}^1$, branched over the three roots z_1, z_2, z_3 of the polynomial $z^3 + pz + q$ and the point ∞.

The differential $\omega = dz/2w$, restricted to E is a holomorphic 1-form (and there is only one such form on an elliptic curve, up to multiplication by constants). Viewed as a C^∞ manifold, the elliptic curve E is homeomorphic to the product of two circles $S^1 \times S^1$, and hence the first homology group $H_1(E, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let the generators of $H_1(E, \mathbb{Z})$ be γ_1 and γ_2. The lattice H is the same as the lattice $\left\{m \int_{\gamma_1} \omega + n \int_{\gamma_2} \omega\right\}$. Indeed, the elliptic integral A_w is determined up to numbers of the form $\int_l \frac{1}{\sqrt{z^3+pz+q}}$, where l is a closed path in $\mathbb{C} \setminus \{z_1, z_2, z_3\}$. On the other hand

$$\int_l \frac{1}{\sqrt{z^3 + pz + q}} = \int_\gamma \omega,$$

where γ is the closed path in E covering l twice.

The integrals $\int_{\gamma_i} \omega$ are called periods of the curve E. The lattice H is called the *period lattice.* The discussion above indicates that the curve E is uniquely determined by its period lattice.

This theory can be extended from elliptic curves (curves of genus 1) to curves of higher genus, and even to higher dimensional varieties.

Let X be a compact Riemann surface of genus g (which is the same as a nonsingular complex projective curve of genus g). It is well known that all Riemann surfaces of genus g are topologically the same, being homeomorphic to the sphere with g handles. They may differ, however, when viewed as complex analytic manifolds. In his treatise on abelian functions (see de Rham [1955]), Riemann constructed surfaces (complex curves) of genus g by cutting and pasting in the complex plane. When doing this he was concerned about the periods of abelian integrals over various closed paths. Riemann called those periods (there are $3g - 3$) *moduli.* These are continuous complex parameters which determine the complex structure on a curve of genus g.

One of the main goals of the present survey is to introduce the reader to the ideas involved in obtaining these kinds of parametrizations for algebraic varieties. Let us explain this in greater detail.

On a Riemann surface X of genus g there are exactly g holomorphic 1-forms linearly independent over $\mathbb{C}$. Denote the space of holomorphic 1-forms on X by $H^{1,0}$, and choose a basis $\omega = (\omega_1, \ldots, \omega_g)$ for $H^{1,0}$. Also choose a basis $\gamma = (\gamma_1, \ldots, \gamma_{2g})$ for the first homology group $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Then the numbers

$$\Omega_{ij} = \int_{\gamma_j} \omega_i$$

are called the *periods* of X. They form the *period matrix* $\Omega = (\Omega_{ij})$. This matrix obviously depends on the choice of bases for $H^{1,0}$ and $H_1(X, \mathbb{Z})$. It turns out (see Chapter 3, Section 1), that the periods uniquely determine the curve X. More precisely, let X and X' be two curves of genus g. Suppose

ω and ω' are bases for the spaces of holomorphic differentials on X and X', respectively, and γ and γ' be are bases for $H_1(X,\mathbb{Z})$ and $H_1(X',\mathbb{Z})$ such that there are equalities

$$(\gamma_i{\cdot}\gamma_j)_X = (\gamma_i'{\cdot}\gamma_j')_{X'}$$

between the intersection numbers of γ and γ'. Then, if the period matrices of X and X' with respect to the chosen bases are the same, then the curves themselves are isomorphic. This is the classical theorem of Torelli.

Now, let X be a non-singular complex manifold of dimension $d > 1$. The complex structure on X allows us to decompose any complex-valued C^∞ differential n-form ω into a sum

$$\omega = \sum_{p+q=n} \omega^{p,q}$$

of components of type (p,q). A form of type (p,q) can be written as

$$\omega^{p,q} = \sum_{(I,J)=(i_1,\ldots,i_p,j_1,\ldots,j_q)} h_{I,J} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

If X is a projective variety (and hence a Kähler manifold; see Chapter 1, Section 7) , then this decomposition transfers to cohomology:

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = \bar{H}^{q,p}. \tag{3}$$

This is the famous Hodge decomposition (Hodge structure of weight n on $H^n(X)$, see Chapter 2, Section 1). It allows us to define the periods of a variety X analogously to those for a curve. Namely, let X_0 be some fixed non-singular projective variety, and $H = H^n(X_0,\mathbb{Z})$. Let X be some other projective variety, diffeomorphic to X_0, and having the same Hodge numbers $h^{p,q} = \dim H^{p,q}(X_0)$. Fix a $\mathbb{Z}$-module isomorphism

$$\phi : H^n(X,\mathbb{Z}) \simeq H.$$

This isomorphism transfers the Hodge structure (3) from $H^n(X,\mathbb{C})$ onto $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$. We obtain the Hodge filtration

$$\{0\} = F^{n+1} \subseteq F^n \subseteq \ldots \subseteq F^0 = H_{\mathbb{C}}$$

of the space $H_{\mathbb{C}}$, where

$$F^p = H^{n,0} \oplus \ldots \oplus H^{p,n-p}, F^{n+1} = \{0\}.$$

This filtration is determined by the variety X up to a $GL(H,\mathbb{Z})$ action, due to the freedom in the choice of the map ϕ. The set of filtrations of a linear space $H_{\mathbb{C}}$ by subspaces F^p of a fixed dimension f^p is classified by the points of the complex projective variety (the *flag manifold*) $F = F(f^n,\ldots,f^1;H_{\mathbb{C}})$. The simplest flag manifold is the Grassmanian $G(k,n)$ of k-dimensional linear subspaces in $\mathbb{C}^n$. The conditions which must be satisfied by the subspaces $H^{p,q}$ forming a Hodge structure (see Chapter 2, Section 1) define a complex submanifold D of F, which is known as the *classifying space* or the *space of period matrices*.

This terminology is easily explained. Let $h^{p,q} = \dim H^{p,q}$. Further, let the basis of $H^{p,q}$ be $\{\omega_j^{p,q}\}$, for $j = 1, \ldots, h^{p,q}$, and let the basis modulo torsion of $H_n(X, \mathbb{Z})$ be $\gamma_1, \ldots, \gamma_b$. Consider the matrix whose rows are

$$I_j^{p,q} = \left(\int_{\gamma_1} \omega_j^{p,q}, \ldots, \int_{\gamma_b} \omega_j p, q \right).$$

This is the period matrix of X. There is some freedom in the choice of the basis elements $\omega_j^{p,q}$, but, in any event, the Hodge structure is determined uniquely if the basis of H is fixed, and in general the Hodge structure is determined up to the action of the group Γ of automorphisms of the $\mathbb{Z}$-module H. Thus, if $\{X_i\}$, $i \in A$ is a family of complex manifolds diffeomorphic to X_0 and whose Hodge numbers are the same, we can define the *period mapping*

$$\Phi : A \to \Gamma \backslash D.$$

We see that we can associate to each manifold X a point of the classifying space D, defined up to the action of a certain discrete group. One of the fundamental issues considered in the present survey is the inverse problem — to what extent can we reconstruct a complex manifold X from the point in classifying space. This issue is addressed by a number of theorems of Torelli type (see Chapter 2, Section 5 for further details).

A positive result of Torelli type allows us, generally speaking, to construct a complete set of continuous invariants, uniquely specifying a manifold with the given set of discrete invariants. Let us look at the simplest example – that of an elliptic curve E. The two-dimensional vector space $H_{\mathbb{C}} = H^1(E, \mathbb{C})$ is equipped with the non-degenerate pairing

$$(\mu, \eta) = \int_E \mu \wedge \eta.$$

Restricting this pairing to $H = H^1(E, \mathbb{Z})$ gives a bilinear form

$$Q_H : H \times H \to \mathbb{Z},$$

dual to the intersection form of 1-cycles on E. We can, furthermore, pick a basis in H, so that

$$Q_H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$H_{\mathbb{C}}$ is also equipped with the Hodge decomposition

$$H_{\mathbb{C}} = \mathbb{C}\omega + \mathbb{C}\bar{\omega},$$

where ω is a non-zero holomorphic differential on E. It is easy to see that

$$\sqrt{-1}(\omega, \bar{\omega}) > 0,$$

and so in the chosen basis $\omega = (\alpha, \beta)$, where

$$\sqrt{-1}(\beta\bar{\alpha} - \alpha\bar{\beta}) > 0. \tag{4}$$

The form ω is determined up to constant multiple. If we pick $\omega = (\lambda, 1)$, then condition (4) means that $\operatorname{Im}\lambda > 0$, and so the space of period matrices D is simply the complex upper half-plane:

$$D = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Now let us consider the family of elliptic curves

$$E_\lambda = \mathbb{C}/\{\mathbb{Z}\lambda + \mathbb{Z}\}, \quad \lambda \in D.$$

This family contains all the isomorphism classes of elliptic curves, and two curves E_λ and $E_{\lambda'}$ are isomorphic if and only if

$$\lambda' = \frac{a\lambda + b}{c\lambda + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Thus, the set of isomorphism classes of elliptic curves is in one-to-one correspondence with the points of the the set $A = \Gamma\backslash D$. The period mapping

$$\Phi : A \to \Gamma\backslash D$$

is then the identity mapping. Indeed, the differential dz defines a holomorphic 1-form in each E_λ.

If γ_1, γ_2 is the basis of $H_1(E_\lambda, \mathbb{Z})$ generated by the elements $\lambda, 1$ generating the lattice $\{\mathbb{Z}\lambda + \mathbb{Z}\}$ then the periods are simply

$$\left(\int_{\gamma_1} \omega, \int_{\gamma_2} \omega\right) = (\lambda, 1).$$

The existence of Hodge structures on the cohomology of non-singular projective varieties gives a lot of topological information (see Chapter 1, Section 7). However, it is often necessary to study singular and non-compact varieties, which lack a classical Hodge structure. Nonetheless, Hodge structures can be generalized to those situations also. These are the so-called *mixed* Hodge structures, invented by Deligne in 1971. We will define mixed Hodge structures precisely in Chapter 4, Section 1, but now we shall give the simplest example leading to the concept of a mixed Hodge structure.

Let X be a complete algebraic curve with singularities. Let S be the set of singularities on X and for simplicity let us assume that all points of S are simple singularities, with distinct tangents. The singularities of X can be resolved by a normalization $\pi : \bar{X} \to X$. Then, for each point $s \in S$ the pre-image $\pi^{-1}(s)$ consists of two points x_1 and x_2, and outside the singular set the morphism

$$\pi : \bar{X}\setminus\pi^{-1}(S) \to X \setminus S$$

is an isomorphism.

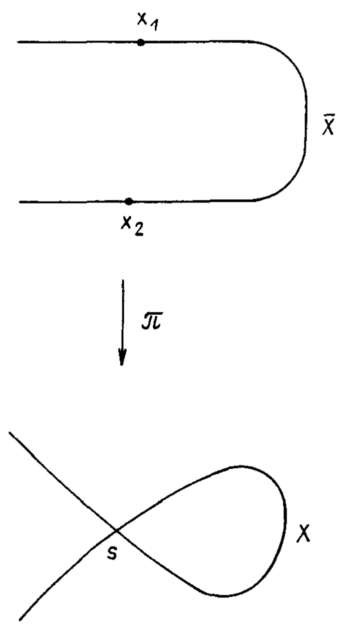


Fig. 1

For a locally constant sheaf $\mathbf{C}_X$ on X we have the exact sequence

$$0 \to \mathbf{C}_X \to \pi_*\mathbf{C}_{\bar{X}} \to \mathbf{C}_S \to 0,$$

which induces a cohomology exact sequence

$$\begin{array}{ccccccccc} 0 & \to & H^0(X,\mathbf{C}_S) & \to H^1(X,\mathbf{C}_X) & \to & H^1(X,\pi_*\mathbf{C}_{\bar{X}}) & \to & 0 \\ & & \wr\| & & & \wr\| & & \\ & & H^0(S,\mathbf{C}_S) & & & H^1(\bar{X},\mathbf{C}_{\bar{X}}) & & \end{array}$$

This sequence makes it clear that $H^1(\bar{X},\mathbf{C}_X)$ is equipped with the filtration $0 \subset H^0(S,\mathbf{C}_S) = W_0 \subset H^1(X,\mathbf{C}_X) = W_1$. The factors of this filtration are equipped with Hodge structures in a canonical way – W_0 with a Hodge structure of weight 0, and W_1/W_0 with a Hodge structure of weight 1, induced by the inclusion of W_1/W_0 into $H^1(\bar{X},\mathbf{C}_X)$.

Even though mixed Hodge structures have been introduced quite recently, they helped solve a number of difficult problems in algebraic geometry – the problem of invariant cycles (see Chapter 4, Section 3) and the description of degenerate fibers of families of of algebraic varieties being but two of the examples. More beautiful and interesting results will surely come.

Here is a brief summary of the rest of this survey.

In the first Chapter we attempt to give a brief survey of classical results and ideas of algebraic geometry and the theory of complex manifolds, necessary for the understanding of the main body of the survey. In particular, the first three sections give the definitions of classical algebraic and complex analytic geometry and give the results *GAGA (Géometrie algébrique et géométrie analytique)* on the comparison of algebraic and complex analytic manifolds.

In Sections 4, 5, and 6 we recall some complex analytic analogues of some standard differential-geometric constructions (bundles, metrics, connections).

Section 7 is devoted to classical Hodge theory.

Sections 8, 9, and 10 contain further standard material of classical algebraic geometry (divisors and line bundles, characteristic classes, extension formulas, Kodaira's vanishing theorem, Lefschetz' theorem on hyperplane section, monodromy, Lefschetz families).

Chapter 2 covers fundamental concepts and basic facts to do with the period mapping, to wit:

Section 1 introduces the classifying space D of polarized Hodge structures and explains the correspondence between this classifying space and a polarized algebraic variety. We study in some depth examples of classifying spaces associated to algebraic curves, abelian varieties and Kähler surfaces. We also define certain naturally arising sheaves on D.

In Section 2 we introduce the complex tori of Griffiths and Weil associated to a polarized Hodge structure. We also define the Abel-Jacobi mapping, and study in detail the special case of the Albanese mapping.

In Section 3 we define the period mapping for projective families of complex manifolds. We show that this mapping is holomorphic and horizontal.

In Section 4 we introduce the concept of variation of Hodge structure, which is a generalization of the period mapping.

In Section 5 we study four kinds of Torelli problems for algebraic varieties. We study the infinitesimal Torelli problem in detail, and give Griffiths' criterion for its solvability.

In Section 6 we study infinitesimal variation of Hodge structure and explain its connection with the global Torelli problem.

In Chapter 3 we study some especially interesting concrete results having to do with the period mapping and Torelli-type results.

In Section 1 we construct the classifying space of Hodge structures for smooth projective curves. We prove the infinitesimal Torelli theorem for non-hyperelliptic curves and we sketch the proof of the global Torelli theorem for curves.

In Section 2 we sketch the proof of the global Torelli theorem for a cubic threefold.

In Section 3 we study the period mapping for K3 surfaces. We prove the infinitesimal Torelli theorem. We construct the modular space of marked K3 surfaces. We also sketch the proof of the global Torelli theorem for K3 surfaces. We study elliptic pencil, and we sketch the proof of the global Torelli theorem for them.

In Section 4 we study hypersurfaces in $\mathbb{P}^n$. We prove the local Torelli theorem, and sketch the proof of the global Torelli theorem for a large class of hypersurfaces.

Chapter 4 is devoted to mixed Hodge structures and their applications.

Section 1 gives the basic definitions and survey the fundamental properties of mixed Hodge structures.

Sections 2 and 3 are devoted to the proof of Deligne's theorem on the existence of mixed Hodge structures on the cohomology of an arbitrary complex algebraic variety in the two special cases: for varieties with normal crossings and for non-singular incomplete varieties.

Section 4 gives a sketch of the proof of the invariant cycle theorem.

Section 5 computes Hodge structure on the cohomology of smooth hypersurfaces in $\mathbb{P}^n$.

Finally, in Section 5 we give a quick survey of some further developments of the theory of mixed Hodge structures, to wit, the period mapping for mixed Hodge structures, and mixed Hodge structures on the homotopy groups of algebraic varieties.

In Chapter 5 we study the theory of degenerations of families of algebraic varieties.

Section 1 contains the basic concepts of the theory of degenerations.

Section 2 gives the definition of the limiting mixed Hodge structure on the cohomology of the degenerate fiber (introduced by Schmid).

In Section 3 we construct the exact sequence of Clemens-Schmid, relating the cohomology of degenerate and non-degenerate fibers of a one-parameter family of Kähler manifolds.

Sections 4 and 5 are devoted to the applications of the Clemens-Schmid exact sequence to the degenerations of curves and surfaces.

In Section 6 we study the degeneration of K3 surfaces. We conclude that the period mapping is an epimorphism for K3 surfaces.

In conclusion, a few words about the prerequisites necessary to understand this survey. Aside from the standard university courses in algebra and differential geometry it helps to be familiar with the basic concepts of algebraic topology (Poincaré duality, intersection theory), homological algebra, sheaf theory (sheaf cohomology and hypercohomology, spectral sequences – see references Cartan–Eilenberg [1956], Godement [1958], Grothendieck [1957], Griffiths–Harris [1978]), theory of Lie groups and Lie algebras (see Serre [1965]), and Riemannian geometry (Postnikov [1971]).

We have tried to either define or give a reference for all the terms and results used in this survey, in an attempt to keep it as self-contained as possible.

Chapter 1
Classical Hodge Theory

§1. Algebraic Varieties

Let us recall some definitions of algebraic geometry.

1.1. Let $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) | z_i \in \mathbb{C}\}$ be the n-dimensional affine space over the complex numbers. *An algebraic set* in $\mathbb{C}^n$ is a set of the form

$$V(f_1, \ldots, f_m) = \{z \in \mathbb{C}^n | f_1(z) = \ldots = f_m(z) = 0\}.$$

where $f_i(z)$ lie in the ring $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ of polynomials in n variables over $\mathbb{C}$. An algebraic set of the form $V(f_1)$ is a *hypersurface* in $\mathbb{C}^n$, assuming that $f_1(z)$ is not a constant.

It is clear that if $f(z)$ lies in the ideal $I = (f_1, \cdots, f_m)$ of $\mathbb{C}[z]$ generated by $f_1(z), \ldots, f_m(z)$ then $f(a) = 0$ for all $a \in V(f_1, \ldots, f_m)$. Thus, to each algebraic set $V = V(f_1, \ldots, f_m)$ we can associate an ideal $I(V) \subset \mathbb{C}[z]$, defined by

$$I(V) = \{f \in \mathbb{C}[z] | f(a) = 0, a \in V\}.$$

The ideal $I(V)$ is a finitely generated ideal, and so by Hilbert's *Nullstellensatz* (Van der Waerden [1971]) $I(V) = \sqrt{(f_1, \ldots, f_m)}$,, where $\sqrt{J} = \{f \in \mathbb{C}[z] | f^k \in J$ for some $k \in \mathbb{N}\}$ is the radical of J.

The ring $\mathbb{C}[V] = \mathbb{C}[z]/I(V)$ is the *ring of regular functions* over the algebraic set V. This ring coincides with the ring of functions on V which are restrictions of polynomials over $\mathbb{C}^n$.

1.2. It is easy to see that the union of any finite number of algebraic sets and the intersection of any number of algebraic sets is again an algebraic set, and so the collection of algebraic sets in $\mathbb{C}^n$ satisfies the axioms of the collection of closed sets of some topology. This is the so-called *Zariski topology.* The Zariski topology in $\mathbb{C}^n$ induces a topology on algebraic sets $V \subset \mathbb{C}^n$, and this is also called the Zariski topology. The neighborhood basis of the Zariski topology on V is the set of open sets of the form $U_{f_1,\ldots,f_k} = \{a \in V | f_1(a) \neq 0, \ldots, f_k(a) \neq 0, f_1, \ldots, f_k \in \mathbb{C}[V]\}$.

Let $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ be two algebraic sets. A map $f : V_1 \to V_2$ is called a *regular mapping* or a *morphism* if there exists a set of m regular functions $f_1, \ldots, f_m \in \mathbb{C}[V_1]$ such that $f(a) = (f_1(a), \ldots, f_m(a))$ for all $a \in V_1$. Obviously a regular mapping is continuous with respect to the Zariski topology. It is also easy to check that defining a regular mapping $f : V_1 \to V_2$ is equivalent to defining a homomorphism of rings $f^* : \mathbb{C}[V_1] \to \mathbb{C}[V_2]$, which transforms the coordinate functions $z_i \in \mathbb{C}[V_2]$ into $f_i \in \mathbb{C}[V_1]$.

Two algebraic sets V_1 and V_2 are called *isomorphic* if there exists a regular mapping $f : V_1 \to V_2$ which possesses a regular inverse $f^{-1} : V_2 \to V_1$.

Alternatively, V_1 and V_2 are isomorphic whenever the rings $\mathbb{C}[V_1]$ and $\mathbb{C}[V_2]$ are isomorphic.

Evidently, for any algebraic set V, the ring of regular functions $\mathbb{C}[V]$ is a finitely generated (over $\mathbb{C}$) algebra. Conversely, if a commutative ring K is a finitely generated algebra over $\mathbb{C}$ without nilpotent elements, then K is isomorphic to $\mathbb{C}[V]$ for some algebraic set V. Indeed, if $z_1, \dots, z_n$ are generators of K, then $K \simeq \mathbb{C}[z_1, \dots, z_n]/I$, where I is the ideal of relations. Thus, $K \simeq \mathbb{C}[V]$, where $V = \{z \in \mathbb{C}^n | f(z) = 0, f \in I\}$. In other words, the category of algebraic sets is equivalent to that of finitely generated algebras over $\mathbb{C}$ without nilpotent elements.

1.3. A product of algebraic sets $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ is the set

$$V \times W = \{(z_1, \dots, z_{n+m}) \in \mathbb{C}^{n+m} | (z_1, \dots, z_n) \in V, (z_{n+1}, \dots, z_{n+m}) \in W\}.$$

It is easy to check that $V \times W$ is an algebraic set, and if $f_i(z_1, \dots, z_n)$, $1 \leq i \leq k$ are generators of $I(V)$ and $g_j(z_1, \dots, z_m)$, $1 \leq j \leq s$ are generators of $I(W)$, then $V \times W$ is defined by the equations $f_i(z_1, \dots, z_n) = 0, g_j(z_{n+1}, \dots, z_{n+m}) = 0$.

1.4. An algebraic set V is called *irreducible* if $I(V)$ is a prime ideal. An algebraic set V is irreducible if V cannot be represented as a union of closed subsets $V_1 \cup V_2$ such that $V \neq V_1, V \neq V_2, V_1 \neq V_2$. It can be shown (Shafarevich [1972]) that every algebraic set is a union of a finite number of irreducible algebraic sets.

If V is an irreducible algebraic set, then $\mathbb{C}[V]$ is an integral domain. Denote the field of quotients of $\mathbb{C}[V]$ by $\mathbb{C}(V)$. This field is called the *field of rational functions* over V, and the transcendence degree of $\mathbb{C}(V)$ over $\mathbb{C}$ is the *dimension* of V, and is denoted by $\dim V$. Elements of $\mathbb{C}(V)$ can be represented as fractions $f(z)/g(z)$ where $f(z), g(z) \in \mathbb{C}(z)$ and $g(z)$ doesn't vanish on all of V. Thus the elements of $\mathbb{C}(V)$ can be viewed as functions defined on a Zariski-open subset of V.

For each point $a \in V$ of an irreducible algebraic set V we define the *local ring* $\mathcal{O}_{V,a} \subset \mathbb{C}(V)$:

$$\mathcal{O}_{V,a} = \left\{ \frac{f}{g} \in \mathbb{C}(V) | f, g \in \mathbb{C}[V], g(a) \neq 0 \right\}.$$

The maximal ideal $m_{V,a} \subset \mathcal{O}_{V,a}$ is

$$m_{V,a} = \left\{ \frac{f}{g} \in \mathbb{C}(V) | f, g \in \mathbb{C}[V], f(a) = 0, g(a) \neq 0 \right\}.$$

In general, for any point a of an arbitrary (not necessarily irreducible) algebraic set V we can also define the local ring as a ring of formal fractions:

$$\mathcal{O}_{V,a} = \left\{ \frac{f}{g} | f, g \in \mathbb{C}[V], g(a) \neq 0 \right\}.$$

with the usual arithmetic operations. Two fractions f_1/g_1 and f_2/g_2 are considered equal if there exists a function $h \in \mathbb{C}[V], h(a) \neq 0$ such that $h(f_1g_2 - f_2g_1) = 0$.

The local rings $\mathcal{O}_{V,a}$ are the stalks of a sheaf of rings $\mathcal{O}_V$ over V, defined as follows. The sections of the sheaf $\mathcal{O}_V$ over an open set $U \subset V$ are fractions $f/g, f, g \in \mathbb{C}[V]$, such that for every $a \in U$ there exists a fraction $f_a/g_a, g_a(a) \neq 0$, which is equal to f/g at a. That is, there exists a function $h_a \in \mathbb{C}[V], h_a(a) \neq 0$, such that

$$h_a(fg_a - f_ag) = 0.$$

This sheaf of rings $\mathcal{O}_V$ is called the *structure sheaf*, and its sections over an open set U are called functions regular over U. Hilbert's *Nullstellensatz* implies that the ring of global sections of $\mathcal{O}_V$ coincides with $\mathbb{C}[V]$.

1.5. To each point $a = (a_1, \ldots, a_n) \in V \subset \mathbb{C}^n$ we associate a linear space called the *tangent space* $T_{V,a}$. The tangent space $T_{V,a}$ is defined to be the subspace of $\mathbb{C}^n$, defined by the system of equations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a)(z_i - a_i) = 0$$

for all $f \in I(V)$. It can be shown that $\dim T_{V,a} \geq \dim V$ for an irreducible V, and furthermore there is a non-empty Zariski-open subset $U \subset V$, such that $\dim T_{V,a} = \dim V$ for all $a \in U$. This set U is defined to be the set of $a \in V$ where the rank of the matrix $\left(\frac{\partial f_i}{\partial z_j}\right)$ is maximal (where $I(V) = (f_1, \ldots, f_m)$).

Let V_i be an irreducible component of an algebraic set V. The points $a \in V_i$ for which $\dim T_{V,a} = \dim V_i$ are called *non-singular* (or *smooth*) points of V.

The tangent space $T_{V,a}$ can be defined an yet another way, as the dual space of the $\mathbb{C}$-linear space $m_{V,a}/m^2_{V,a}$. Indeed, for every function $h = f(z)/h(z) \in \mathcal{O}_{V,a}$ define the differential

$$d_ah = \sum_{i=1}^{n} \frac{\partial h}{\partial z_i}(z_i - a_i).$$

This differential satisfies the conditions

$$d_a(h_1 + h_2) = d_ah_1 + d_ah_2 \tag{1}$$

and

$$d_a(h_1h_2) = h_1(a)d_ah_2 + h_2(a)d_ah_1. \tag{2}$$

Since $d_a(c) = 0$ for c a constant function, the differential d_a is actually determined by its values on $m_{V,a}$. For every $h \in m_{V,a}$ d_ah determines a linear function $d_ah : T_{V,a} \to \mathbb{C}$. From equation (2) it follows that $d_ah = 0$ for any $h \in m^2_{V,a}$. Thus d_a defines a mapping $d_a : m_{V,a}/m^2_{V,a} \to T^*_{V,a}$. This map is easily checked to be an isomorphism.

Let $V \in \mathbb{C}^n$. Consider an algebraic set $T_V \in \mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ defined by the equations

$$f(z_1, \dots z_n) = 0,$$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z_1, \dots, z_n)(z_{i+n} - z_i) = 0,$$

for $f \in I(V)$. Let π be the projection map $\pi : T_V \to V$, where $\pi(z_1, \dots, z_{2n}) = (z_1, \dots, z_n)$. Evidently $\pi(T_V) = V$ and $\pi^{-1}(a) = T_{V,a}$ for any $a \in V$. Thus T_V fibers over V, with fibers being just the tangent spaces at the points $a \in V$. The algebraic set T_V is the *tangent bundle* to V.

1.6. Algebraic Varieties The concept of *algebraic variety* is central to algebraic geometry, and there are several ways to define this. The most general approach is that of Grothendieck (see Shafarevich [1972], Hartshorne [1977]), where an algebraic variety is defined to be a reduced separable scheme of finite type over a field k. Since we will not need such generality, we will follow A. Weil, and define an algebraic variety to be a ringed space, glued together from algebraic sets. Recall that a *ringed space* is an ordered pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a family of ring homomorphisms $f_U^* : \mathcal{O}_Y|U \to \mathcal{O}_X|f^{-1}(U)$ for all open sets $U \subset Y$, which agree on intersections of open sets.

An *affine variety* is a ringed space $(V, \mathcal{O}_V)$ where V is an algebraic set and $\mathcal{O}_V$ is its structure sheaf. Note that for an affine variety V, the open sets (which are a neighborhood basis in the Zariski topology) of the form

$$U_f = \{z \in V | f(z) \neq 0\},$$

where f is a function regular on V are affine varieties. Indeed, if $V \subset \mathbb{C}^n$, then U_f is isomorphic to the algebraic set in $\mathbb{C}^{n+1}$ defined by the equations $z_{n+1} f(z_1, \dots, z_n) = 1$ and $f_i(z_1, \dots, z_n) = 0$, where $f_i(z) \in I(V) \subset \mathbb{C}[z_1, \dots, z_n]$.

Definition. A ringed space $(X, \mathcal{O}_X)$ is an *algebraic variety* if X can be covered by a finite number of open everywhere-dense sets V_i, so that $(V_i, \mathcal{O}_X|V_i)$ are isomorphic to affine varieties and X is separable: the image of X under the diagonal embedding $\Delta = (\mathrm{id}, \mathrm{id}) : X \to X \times X$ is closed in $X \times X$. (The definition of a product of affine algebraic sets can be naturally extended to ringed spaces).

Example Projective space $\mathbb{P}^n$. Let $\mathbb{P}^n$ be the set of all the lines through the origin in $\mathbb{C}^{n+1}$. Let us give $\mathbb{P}^n$ the structure of an algebraic variety. To do this, note that a line $l \subset \mathbb{C}^{n+1}$ is uniquely determined by a point $u = (u_0, \dots, u_n) \in l, u \neq 0$. The points u and $\lambda u = (\lambda u, \dots, \lambda u_n)$ define the same line. Thus

$$\mathbb{P}^n = \left\{u \in \mathbb{C}^{n+1}\right\} \setminus \{0\} \,/\, (u \sim \lambda u, \lambda \neq 0).$$

The coordinates $(u_0, \ldots, u_n)$ are the *homogeneous coordinates* for $\mathbb{P}^n$. The set U_i of $\mathbb{P}^n$ for which $u_i \neq 0$ can be naturally identified with $\mathbb{C}^n$ by means of the mapping $\phi_i : U_i \to \mathbb{C}^n$:

$$\phi_i(u_0, \ldots, u_n) = \left(\frac{u_0}{u_i}, \ldots, \frac{\widehat{u_i}}{u_i}, \ldots, \frac{u_n}{u_i}\right) = (z_1, \ldots, z_n) \in \mathbb{C}^n .$$

The transition function between U_i and U_j is given by

$$\phi_j \circ \phi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{\widehat{z_j}}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_n}{z_j}\right),$$

and all of the functions z_k/z_j and $1/z_j$ are rational functions on $\mathbb{C}^n = U_i$, regular on $U_i \cap U_j$. This allows us to view $\mathbb{P}^n$ as an algebraic variety.

Closed subsets of $\mathbb{P}^n$ are sets of the type

$$V_{f_1,\ldots,f_k} = \{u = (u_0, \ldots, u_n) \in \mathbb{P}^n | f_i(u_0, \ldots, u_n) = 0, 1 \leq i \leq k\},$$

where $f_i(u_0, \ldots, u_n)$ are homogeneous polynomials. The intersection $V_{f_1,\ldots,f_k} \cap U_j$ is given in $U_j = \mathbb{C}^n$ by the equations

$$f_i\left(\frac{u_0}{u_j}, \ldots, \frac{\widehat{u_j}}{u_j}, \ldots, \frac{u_n}{u_j}\right) = f_i(z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n) = 0,$$

hence is an affine variety. Thus, closed sets in $\mathbb{P}^n$ are algebraic varieties. An algebraic variety isomorphic to a closed sub-variety of $\mathbb{P}^n$ is called a *projective variety.*

1.7. Let us extend the definition of a field of rational functions from affine algebraic sets to general algebraic varieties. First, note that if an affine variety V is irreducible and $U \subset V$ is an open affine sub-variety of V, then U is also irreducible, and furthermore, the restriction to U of rational functions defined on V is an isomorphism of fields $\mathbb{C}(U)$ and $\mathbb{C}(V)$. Thus, if U_1 and U_2 are non-empty affine open subsets of an irreducible algebraic variety X, then there are natural isomorphisms $\mathbb{C}(U_1) \simeq \mathbb{C}(U_1 \cap U_2) \simeq \mathbb{C}(U_2)$. Similarly we can define the field of rational functions $\mathbb{C}(X)$ on an irreducible algebraic variety X. The elements of $\mathbb{C}(X)$ are rational functions f_U defined on non-empty affine open sub-varieties $U \subset X$, where $f_{U_1} = f_{U_2}$ if the restrictions of f_{U_1} and of f_{U_2} to $U_1 \cap U_2$ agree.

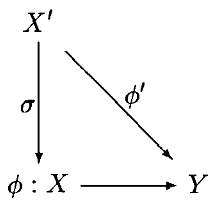
The concept of rational function can be generalized to that of a *rational mapping* between algebraic varieties. A rational mapping $\phi : X \to Y$ of algebraic varieties is an equivalence class of pairs (U, ϕ_U), where U is a non-empty open subset of X while ϕ_U is a morphism from U to Y. Two pairs (U, ϕ_U) and (V, ϕ_V) are considered equivalent, if ϕ_U and ϕ_V agree on $U \cap V$. For any rational mapping we can choose a representative $(\tilde{U}, \phi_{\tilde{U}})$, such that $U \subset \tilde{U}$ for any equivalent pair (U, ϕ_U). The open set $\tilde{U}$ is called the *domain of definition of the rational mapping.* If $\phi_{\tilde{U}}$ is everywhere dense in Y, then the rational mapping ϕ defines an inclusion of fields $\phi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ (if X and Y are irreducible). If ϕ^* is an isomorphism, then X and Y are said to be *bi-rationally isomorphic*. In other words, X and Y are birationally isomorphic, if there is an open dense subsets U_X and U_Y, which are isomorphic to one another.

One of the most important examples of bi-rational isomorphism is the *monoidal transformation centered on a smooth sub-variety*, which can be defined as follows. Let X be a non-singular algebraic variety, $\dim X = n$,and $C \subset X$ is a non-singular algebraic sub-variety, $\dim C = n - m$. The X can be covered by affine neighborhoods $U_k \subset X$, where C is defined by the equations $u_{k,1} = \ldots = u_{k,m} = 0$, where $u_{i,k}$ are regular in U_k and $u_{k,1}, \ldots, u_{k,m}$ generate the ideal $I(C \cap U_k)$ in $\mathbb{C}[U_k]$ (see Shafarevich [1972]). Consider a sub-variety U_k' of $U_k \times \mathbb{P}^{m-1}$ defined by the equations

$$u_{k,i} \cdot t_j = u_{k,j} \cdot t_i, \quad 1 \leq i, j \leq m,$$

where $(t_1, \ldots, t_m)$ are homogeneous coordinates in in $\mathbb{P}^{m-1}$ and let σ_k be the restriction of the projection map $p_1 : U_k \times \mathbb{P}^{m-1} \to U_k$ to C. It is easy to see that $\sigma^{-1}(x)$ is isomorphic to $\mathbb{P}^{m-1}$ for every $x \in C$ and for $x \notin C$ $\sigma_k^{-1}(x)$ is a single point, so δ defines an isomorphism between $U_k' \setminus \sigma^{-1}(C)$ and $U_k \setminus C$.It is also easy to check that the variety $U_k' \subset U_k \times \mathbb{P}^{m-1}$ doesn't depend on the choice of the equations defining the subvariety C in U_k. Therefore, the varieties U_k' can be glued together into a single variety X', and thus to obtain a morphism $\sigma : X' \to X$, such that $\sigma^{-1}(x) = \mathbb{P}^{m-1}$ for every $x \in C$ and $\sigma : X' \setminus \sigma^{-1}(C) \to X \setminus C$ is an isomorphism. The resulting map σ is called the monoidal transformation of the variety X centered on C.

Let $\phi : X \to Y$ be a rational mapping of non-singular algebraic varieties. Then, according to a theorem of Hironaka [1964], we can resolve the points where ϕ is undefined by a sequence of monoidal transformations with non-singular centers. That is, there is a commutative diagram in which σ is a composition of monoidal transformations with non-singular centers, while ϕ' is a morphism.



§2. Complex Manifolds

2.1. Let us equip $\mathbb{C}^n$ with a topology whose neighborhood basis consists of polydisks $\Delta^n_{a,\epsilon}$, of radius $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, centered at $a \in \mathbb{C}^n$:

$$\Delta^n_{a,\epsilon} = \{z \in \mathbb{C}^n \,||z_i - a_i| < \epsilon_i\}.$$

We will refer to the topology defined above as the *complex topology*.

Recall that a complex-valued function $f(z)$, defined in some neighborhood U_a of $a \in \mathbb{C}^n$, is called *analytic* (or *holomorphic*) at a if there exists a polydisk $\Delta^n_{a,\epsilon} \subset U_a$, in which f can be represented as a convergent power series:

$$f(z) = \sum_{\alpha \geq 0} c_\alpha (z-a)^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, and $(z-a)^\alpha = (z_1 - a_1)^{\alpha_1} \ldots (z_n - a_n)^{\alpha_n}$.

Denote by $\mathcal{O}_{n,a}$ the subring of the ring of formal power series $\mathbb{C}[[z-a]]$ at a, consisting of those $f \in \mathbb{C}[[z-a]]$ which converge in some neighborhood $U_{(f)}$ of $a \in \mathbb{C}^n$. It can be checked that $\mathcal{O}_{n,a}$ is a Noetherian local ring with unique factorization. The unique maximal ideal of $\mathcal{O}_{n,a}$ consists of the analytic functions vanishing at a. The ring $\mathcal{O}_{n,a}$ is called the ring of *germs of analytic functions* at a.

2.2. A subset $V \subset \mathbb{C}^n$ is called *analytic,* if for any $a \in V$ there exists a neighborhood U_a such that $V \cap U_a$ coincides with a zero set of a finite set of functions analytic at a. In particular, every algebraic set $V \subset \mathbb{C}_n$ is analytic.

Let f be a function defined on an analytic set V. We say that f is *analytic* at $a \in V$, if there exists a neighborhood $U_a \in V$, where f is a restriction to V of a function $F \in \mathcal{O}_{n,a}$. Just as we did for algebraic sets, we can define a local ring $\mathcal{O}_{V,a}$ of germs of functions on V analytic at a. That is, $\mathcal{O}_{V,a} = \mathcal{O}_{n,a}/I_a(V)$, where $I_a(V)$ is the ideal of functions in $\mathcal{O}_{n,a}$ which vanish on V on some neighborhood of a. The rings $\mathcal{O}_{V,a}$ can be glued into a sheaf $\mathcal{O}_V$ of functions holomorphic on V. The sections of $\mathcal{O}_V$ over an open set $U \subset V$ are functions analytic at every point $a \in U$.

A continuous mapping $\phi : V_1 \to V_2$ of analytic sets is called a *holomorphic mapping* if for every point $a \in V_1$ and every function f analytic at $\phi(a)$, the function $\phi \circ f$ is analytic at a. The holomorphic map $\phi : V_1 \to V_2$ is an *isomorphism* if there exists a holomorphic inverse ϕ^{-1}.

The tangent space $T_{V,a}$ to $V \subset \mathbb{C}^n$ at a is defined by the equations

$$\sum_{i=1}^n \frac{\partial f}{\partial z_i}(a)(z_i - a_i) = 0,$$

$$f \in I_a(V),$$

analogously to the algebraic situation. Also analogously, $T_{V,a} \simeq (m_{V,a}/m^2_{V,a})^*$, where $m_{V,a}$ is the maximal ideal of the ring $\mathcal{O}_{V,a}$. Just as in the algebraic situation, the tangent spaces $T_{V,a}$ can be glued together to mke the tangent bundle $T_V \subset \mathbb{C}^{2n}$, and there exists a projection map $\pi : T_V \to V$, such that $\pi^{-1}(a) = T_{V,a}$.

A holomorphic mapping $\phi : V_1 \to V_2,$, $\phi(a) = b \in V_2$, induces a map $\phi_* : T_{V,a} \to T_{V,b}$ as follows. By definition, ϕ induces $\phi^* : \mathcal{O}_{V_2,b} \to \mathcal{O}_{V_1,a}$, such that $\phi^*(f) = \phi \circ f$. It is easy to see that $\phi^*(m_{V_2,b}) \subset m_{V_1,a}$ and $\phi^*(m^2_{V_2,b}) \subset m^2_{V_1,a}$. Therefore, we can define a map